

# Relaxation times in an open interacting two-qubit system

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In a two-qubit system the coupling with an environment considerably affects the entanglement dynamics, usually leads to the loss of entanglement within a finite time. Since entanglement is a key feature in the application of such systems to quantum information processing, it is highly desirable to find a way to prolonging its lifetime. We present a simple model of an interacting two-qubit system in the presence of a thermal Markovian environment. The qubits are modeled as interacting spin-1/2 particles in a magnetic field and the environment is limited to inducing single spin-flip events. A simple scheme allows us to calculate the relaxation rates for all processes. We show that the relaxation dynamics of the most entangled state exhibit critical slowing down as a function of the magnetic field, where the relaxation rate changes from exponentially small values to finite values in the zero-temperature limit. We study the effect of temperature and magnetic field on all the other relaxation rates and find that they exhibit unusual properties, such as nonmonotonic dependence on temperature and a discontinuity as a function of magnetic field. In addition, a simple scheme to include non-Markovian effects is presented and applied to the two-qubit model. We find that the relaxation rates exhibit a sharp, cusplike resonant structure as a function of the environment memory time, and that for long memory times all the different relaxation rates merge into a single one.

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## I. INTRODUCTION

In consideration of the application of quantum information processing, two main ingredients must be considered—entanglement and decoherence. While entanglement, or non-local coherence, plays a key role in qubit operations [1], decoherence sets the limit to which such operations may be performed [2]. Decoherence may result from the interaction of the quantum system with a dissipative environment [3], dramatically affecting the dynamics of the quantum system, and its entanglement properties. Specifically, it was theoretically predicted [4–6] that a (noninteracting) two-qubit system can be completely disentangled in a finite time, a phenomenon dubbed “entanglement sudden death.” This was followed by a plethora of theoretical studies of this phenomenon in various situations, most of them treating noninteracting qubits (i.e., a pair of qubits which interact with each other only in the mediation of the environment), with either a Markovian or non-Markovian environment [7–17]. From the experimental side, direct measurements of entanglement have been performed [18,19], and the entanglement sudden death was observed [20].

The loss of entanglement seems to be a generic feature of two-qubit systems [6]. Our goal is to study a simple system where entanglement sudden death may be avoided. For this aim we study a simple model of an interacting two-qubit system in the presence of a thermal dissipative bath. We model the qubits as interacting spin- $\frac{1}{2}$  particles in a magnetic field [21] and use the Born-Markov approximation for the system-environment coupling. The environment is assumed to be Ohmic and induce thermal transitions, and it allows for only single spin-flip events (in similarity to spin-boson models [22]). We calculate analytically the full dynamics of this system, with emphasis given to the different relaxation rates. We point out here that we use the term “relaxation rates” loosely, to describe the time-scales of both processes which

include energy change (relaxation) and only coherence loss (decoherence). Both types of processes are inherently present in our calculation scheme. We show that, although the coupling with the environment is characterized by a single relaxation rate, different relaxation rates emerge for different coherent states. We study the effect of temperature and magnetic field on the different relaxation rates and find that they may be nonmonotonic functions of temperature.

As the main results of this paper, we demonstrate that as a function of the magnetic field the relaxation rate of the highly entangled states abruptly changes from being finite as the temperature vanishes to being exponentially small. This occurs when one of the states of the entangled pair is in a metastable state, and indicates that with a proper tuning of parameters the entanglement may survive for very long times even in the presence of a dissipative environment. We demonstrate the long life of entanglement by calculating the concurrence of a specific entangled state, and show that application of a transverse magnetic field destroys this effect.

Finally, we devise a simple way to account for non-Markovian effects when calculating relaxation rates. Studying the relaxation rates as a function of the environment memory time, we find that some relaxation rates exhibit a nonmonotonic dependence on the memory time, with a cusplike resonance. For long memory times, the different relaxation rates merge into a single rate.

## II. METHOD

Let us introduce our method for calculating the relaxation times. We consider a quantum system, characterized by a time-independent Hamiltonian  $\mathcal{H}$  with  $N$  energy levels  $E_k$ ,  $k=1, 2, \dots, N$ . The system dynamics are given by the evolution of the density matrix  $\rho(t) = \sum_{kk'} \rho_{kk'}(t) |k\rangle\langle k'|$ , where  $|k\rangle$  are the eigenfunctions of the Hamiltonian [23]. For the above choice of Hamiltonian, in the Markovian approximation the

evolution of the density matrix is given by a quantum master equation [24] ( $\hbar=1$ ),

$$\dot{\rho}(t) = -i[\mathcal{H}, \rho] + \mathcal{L}\rho, \quad (1)$$

where  $\mathcal{L}\rho$  is a superoperator describing the dissipative dynamics. It is commonly taken in the Lindblad form [25]

$$\mathcal{L}\rho = \sum_i \left( -\frac{1}{2}\{V_i^\dagger V_i, \rho\} + V_i \rho V_i^\dagger \right) \quad (2)$$

(where  $\{\cdot, \cdot\}$  are the anticommutation relations). This form is the simplest form for the quantum master equation that defines the evolution of the density matrix, while conserving both the Hermiticity of the density matrix, its trace (equal to unity), and ensuring its positivity [26]. The  $V$  operators define the different relaxation processes induced by the environment.

We now follow Ref. [27] and cast the density matrix into a vector form, defining  $\tilde{\rho} = (\rho_{11}, \rho_{22}, \dots, \rho_{NN}, \text{Re } \rho_{12}, \text{Im } \rho_{12}, \dots, \text{Re } \rho_{1N}, \text{Im } \rho_{1N}, \dots)$ . Here the first  $N$  elements account for occupation probabilities and the other elements describe coherence between the states in the statistical mixture. It is now a matter of rearranging the master equation into a form  $\dot{\tilde{\rho}}(t) = \hat{M}\tilde{\rho}(t)$ , where now  $\hat{M}$  is a matrix of dimension  $N^2$  which includes both the Hamiltonian and the dissipative part of the evolution.

Due to the semigroup properties of the Lindblad equation, (at least) one of the eigenvalues of  $\hat{M}$  is exactly zero [27]. The eigenvector corresponding to this eigenvalue is the steady state of the system, i.e., the limit  $\lim_{t \rightarrow \infty} \rho(t)$ . The other eigenvalues may have an imaginary part, but all of them have a negative real part, which corresponds to the relaxation rate of the corresponding eigenvector. Thus, by calculating the eigenvalues of  $\hat{M}$  one obtains the relaxation rates for all possible processes. For the most general initial condition, the smallest nonvanishing eigenvalue of  $\hat{M}$  (which we call  $\lambda_1$ ) represents the longest relaxation rate.

Two remarks are in order here. The first is that it is not surprising that the real parts of the eigenvalues of  $\hat{M}$  are negative. This stems from the fact that  $\hat{M}$  is only a recasting of the original Lindblad operator, the positivity of which is ensured by the Lindblad theorem [25,26]. The second is that in our calculations we always obtain a single steady state (i.e., a single eigenvalue of  $\hat{M}$  that is equal to zero). Indeed, in the general case there may be several steady states, and in general there is no *a priori* way of knowing how many steady states exist. However, our simple model gives a physically transparent reason for this. Consider our system at zero temperature. Then, the relaxation operators simply transfer high-energy states into lower-energy states, with the inevitable outcome that the system will end up in the ground state. Since the ground state is nondegenerate in our case (due to the magnetic field), there is a unique steady-state solution. Applying a finite temperature simply generates a solution which is a statistical combination of the ground state and the excited states (with the appropriate Boltzmann weights), but still, since there is only a single ground state, this “thermodynamic” state is unique.

### III. APPLICATION TO THE TWO-QUBIT SYSTEM

#### A. Two-qubit system in a perpendicular field

Next we consider the application of the above method to our model two-qubit system. The Hamiltonian, with Ising interaction, can be written as

$$\mathcal{H} = -2J s_z^{(1)} s_z^{(2)} + \mathbf{B} \cdot \sum_{i=1,2} \mathbf{s}^{(i)}, \quad (3)$$

where  $\mathbf{s}^{(i)}$  are the two qubit levels (i.e., spin= $\frac{1}{2}$  particles),  $J>0$  describes the interactions, and  $\mathbf{B}$  is the external magnetic field. This choice of Hamiltonian is not only convenient (being very simple), but also represents several suggestions for realistic, spin-based quantum computers [28,29].

For simplicity we start with magnetic field only in the  $z$  direction, i.e.,  $\mathbf{B} = B\mathbf{z}$  with  $B>0$ . Choosing as a basis the four states  $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$  (which we number from 1 to 4, respectively), the Hamiltonian can be written (up to a constant energy shift) as

$$\mathcal{H} = \begin{pmatrix} -J-B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -J+B \end{pmatrix}. \quad (4)$$

In order to account for the environment, the spins are coupled to one that induces spin-flip processes, represented by the  $V$  operators in Eq. (2). Here we make two assumptions, namely, (i) the spins are flipped one at a time (i.e., there is no direct relaxation from the  $|\uparrow\uparrow\rangle$  state to the  $|\downarrow\downarrow\rangle$  state, etc.) and (ii) the relaxation rate between two states is proportional to the Boltzmann factor of the corresponding energy difference between the states [24] (these two assumptions on the form of the  $V$  operators reflect the properties of the environment as described in the Introduction). For example, the relaxation operator from  $|\uparrow\uparrow\rangle$  to  $|\uparrow\downarrow\rangle$  (taking  $k_B=1$ ) is [30]

$$V_{12} = \gamma_{12}^{1/2} |\uparrow\downarrow\rangle \langle \uparrow\uparrow|, \\ \gamma_{12} = \frac{\gamma_0}{\cosh\left(\frac{J+B}{2T}\right)} \exp\left(-\frac{J+B}{2T}\right), \quad (5)$$

where  $\gamma_0$  is some typical relaxation rate which represents the strength of the qubit-environment coupling. For the reverse process the relaxation operator is

$$V_{21} = \gamma_{21}^{1/2} |\uparrow\uparrow\rangle \langle \uparrow\downarrow|, \\ \gamma_{21} = \frac{\gamma_0}{\cosh\left(\frac{J+B}{2T}\right)} \exp\left(\frac{J+B}{2T}\right), \quad (6)$$

so that the condition of detailed balance is maintained, i.e.,  $\gamma_{12}/\gamma_{21} = \exp(-\Delta E_{12}/T)$ . Note that one can normalize the transition rate in different ways and still maintain detailed balance. Here we choose a normalization that keeps all relaxation rates finite (even at  $T \rightarrow 0$ ) and preserves the  $B \rightarrow$

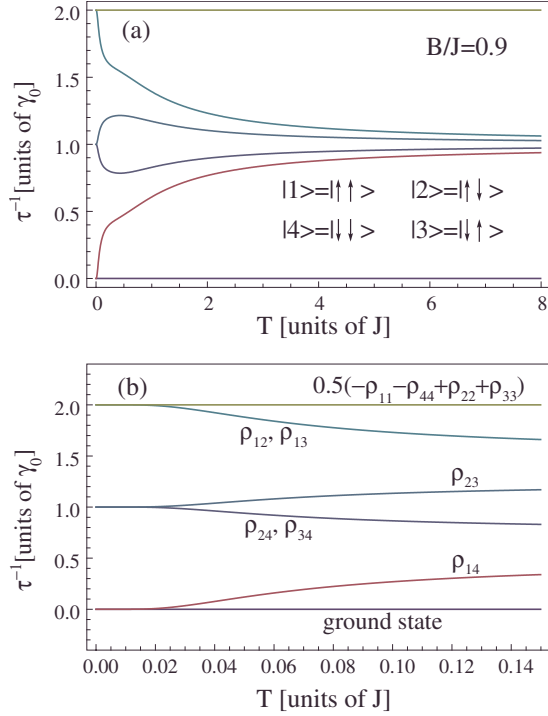


FIG. 1. (Color online) Different relaxation rates (at  $B/J=0.9$ ) as a function of temperature (a) for a large temperature scale, and (b) zoomed in on the low-temperature regime. For each rate, the corresponding density matrix element is noted.

$-B$  symmetry (i.e., does not depend on the gauge of the Hamiltonian). However, all the qualitative results presented in this paper equally apply for a different choice of the normalization of the  $V$  operators.

Once the form of the  $V$  operators is specified, it is now a matter of algebraic manipulation to obtain the  $\hat{M}$  matrix and its eigenvalues and eigenvectors (the full form of  $\hat{M}$  is given in the Appendix). For the steady state we find

$$\rho(\infty) = Z^{-1} [ |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + e^{-2B/T} |\downarrow\downarrow\rangle\langle\downarrow\downarrow| + e^{-(J+B)/T} ( |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| ) ], \quad (7)$$

with  $Z$  being the partition function of the system, which results in a pure  $|\uparrow\uparrow\rangle$  state (which is the ground state of the Hamiltonian) in the limit of  $T \rightarrow 0$ .

For the above example, all the eigenvalues may be calculated analytically. For the lowest rate (i.e., minus the real part of the eigenvalues) we find

$$\lambda_1 = 1 - \frac{\sinh(J/T)}{\cosh(B/T) + \cosh(J/T)}, \quad (8)$$

which is three-fold degenerate. Two states contain  $\rho_{14}$  and  $\rho_{41}$  (i.e., a coherence between  $|\uparrow\uparrow\rangle$  and  $|\downarrow\downarrow\rangle$ ), and the third is a mixture of all the diagonal elements  $\rho_{11}$ ,  $\rho_{22}$ ,  $\rho_{33}$ , and  $\rho_{44}$ . Inspection of  $\lambda_1$  in the limit  $T \rightarrow 0$  shows that

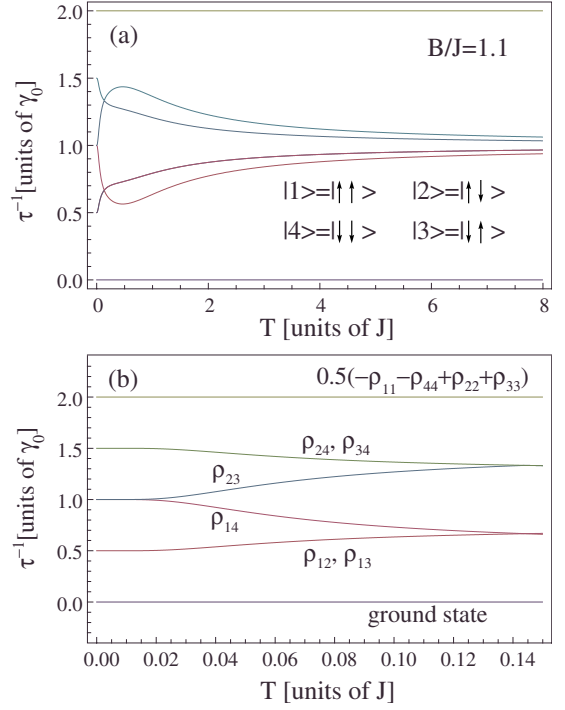


FIG. 2. (Color online) Same as in Fig. 1 but for  $B/J=1.1$ .

$$\lim_{T \rightarrow 0} \lambda_1 = \begin{cases} 0, & B < J, \\ \gamma_0/2, & B = J, \\ \gamma_0, & B > J. \end{cases} \quad (9)$$

This means that for  $B < J$  the relaxation time from the coherent  $\rho_{14}$  state diverges, i.e., the system *never* reaches its disentangled ground state, and the latter is thus the only nondegenerate steady state, as we have previously discussed. However, for  $B \geq J$  it relaxes to the ground state on a time scale  $\tau \sim \gamma_0^{-1}$ . This can be easily explained from the following considerations. Note that the  $|\downarrow\downarrow\rangle$  state can relax only into one of the degenerate middle states,  $|\uparrow\downarrow\rangle$  or  $|\downarrow\uparrow\rangle$ . For  $0 < B < J$ , its energy is negative (but higher than the ground state energy), and therefore the relaxation rate is exponentially small. Put differently, the  $|\downarrow\downarrow\rangle$  state is a metastable state, escape from which requires an exponentially rare correlated event. For  $B > J$ , however, the energy of  $|\downarrow\downarrow\rangle$  is no longer negative, and hence it can relax into the ground state by a cascade relaxation through the middle states.

While all the other eigenvalues are available analytically, writing them in full form is cumbersome, and thus we present them graphically. In Fig. 1 we plot the relaxation rates for the different states as a function of temperature for magnetic field  $B/J=0.9$ . In Fig. 1(a) we plot a wide temperature range, and we zoom in on the low-temperature range in Fig. 1(b). For each relaxation rate the corresponding elements of the density matrix are marked. Figure 1 shows two interesting features: (i) the relaxation rates are nonmonotonic in temperature, and (ii) they break into groups, with only two possible time scales ( $\gamma_0^{-1}$  and  $2\gamma_0^{-1}$ ) at  $T \rightarrow 0$ .

For comparison, in Fig. 2 we plot the same for a magnetic field  $B/J=1.1$ . We now find that the relaxation rates are bro-

ken into four groups, i.e., additional time scales appear. Interestingly, only the  $\rho_{23}$  state preserves its  $T \rightarrow 0$  limit as the  $B=J$  point is crossed, while all other time scales exhibit a discontinuous change.

### B. Concurrence

In order to demonstrate the above effect on the entanglement of the two-qubit system, we calculate the concurrence, which is a direct measure of the entanglement [31]. For the two-qubit system (in the basis chosen above) it is defined as  $C = \max(0, \sqrt{e_1} - \sqrt{e_2} - \sqrt{e_3} - \sqrt{e_4})$ , where  $e_i$  are the eigenvalues of the matrix  $\hat{\zeta}$ , defined by  $\hat{\zeta} = \rho(s_y^{(1)} \otimes s_y^{(2)}) \rho^*(s_y^{(1)} \otimes s_y^{(2)})$ . Clearly, the dynamics of the concurrence depend on the initial condition. For this example we choose the initial density matrix [11]

$$\rho(0) = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}, \quad (10)$$

for which one can easily calculate the concurrence  $C(t) = \max[0, 2(\sqrt{\rho_{14}\rho_{41}} - \sqrt{\rho_{22}\rho_{33}})]$ . The diagonal elements are given by Eq. (7) and  $\rho_{14} = \rho_{41} = 1/2$ . By making this choice we start with a highly entangled state ( $C \approx 1$ ), but the dynamics are very easy to calculate as the diagonal elements do not change at all, and the off-diagonal ones decay with the rate given by Eq. (9). We point out that one can start with any initial diagonal elements and obtain results similar to those presented below, since the diagonal elements quickly relax to the steady state [Eq. (7)] and do not contribute to the concurrence time evolution.

In Fig. 3 we plot the concurrence as a function of temperature and time for magnetic field values  $B/J=0.9$  (upper panel) and  $B/J=1.1$  (lower panel). The mesh corresponds to finite  $C$ , while in the blank regions  $C=0$ . For  $B > J$  we find that entanglement sudden death is present at all temperatures. However, for  $B < J$  it becomes exponentially suppressed at lower temperatures: the concurrence remains practically finite for all times.

In order to understand this behavior, we note that the formula for the concurrence describes a competition between the off-diagonal elements of the density matrix, which contribute to the entanglement, and the diagonal elements, which disentangle the state. At strictly  $T=0$ , the diagonal elements vanish but the off-diagonal elements survive indefinitely, giving rise to an entangled state. For very low temperature, while in the strict  $t \rightarrow \infty$  limit the system becomes disentangled, this time is exponentially long.

### C. Transverse field

Let us consider the effect of an additional transverse field  $\mathbf{B} = B_x \mathbf{x}$  (the results are identical to those for an additional field in the  $y$  direction). A transverse field can represent either an inherent interaction between the two qubit states, or an actual field (in the qubit relevant Hilbert space), which is used to perform quantum operations. Since both these ingre-

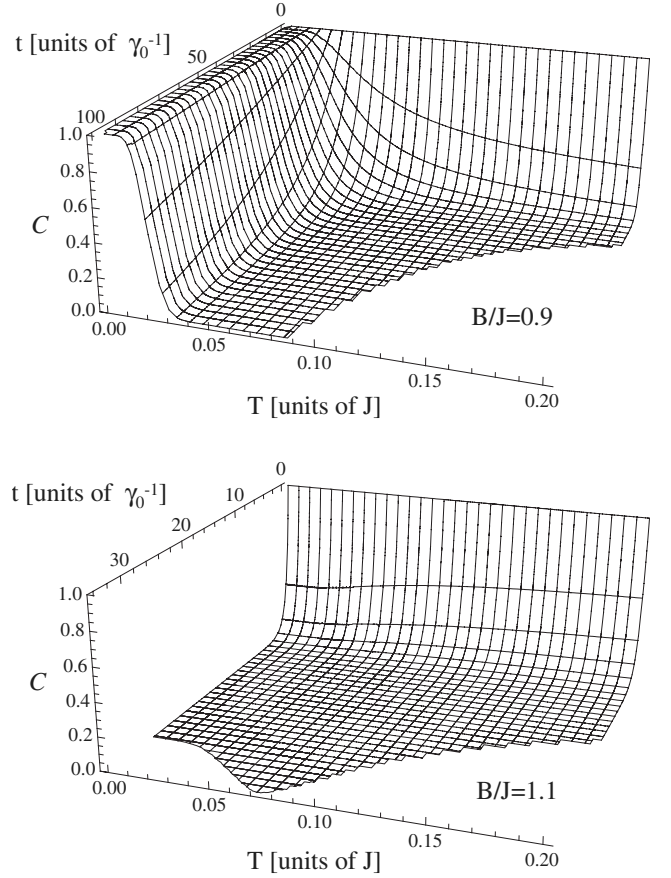


FIG. 3. Concurrence  $C$  as a function of temperature and time for magnetic field values  $B/J=0.9$  (upper panel) and  $B/J=1.1$  (lower panel). The mesh corresponds to finite  $C$ , while in the blank regions  $C=0$ . For  $B < J$  at low temperatures the concurrence practically never decays, while for  $B > J$  sudden death of entanglement appears for every temperature.

dients appear in any implementation of a physical qubit, it is important to study their effect on the relaxation time scales.

We thus repeat the above procedure of constructing and diagonalizing the  $\hat{M}$  matrix with an additional term. In Fig. 4 we plot the inverse relaxation rates at  $T=0$ ,  $B/J=0.9$  as a function of  $B_x$ . It is found that the infinite relaxation time be-

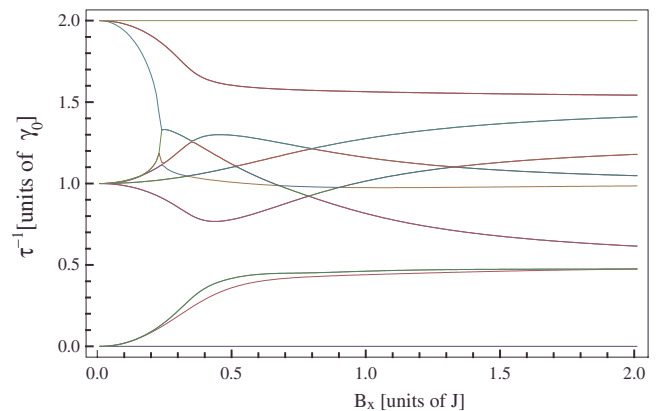


FIG. 4. (Color online) Relaxation rates (at  $T=0$ ,  $B/J=0.9$ ) as a function of perpendicular magnetic field  $B_x$ .



comes finite (i.e., there is no longer a metastable state) and that the degeneracy is lifted. This means that applying a transverse field may give rise to entanglement sudden death. Interestingly, one can see that the time scales are not monotonic functions of  $B_x$ .

The conclusion arising from the above calculation is two-fold: (i) in a realistic qubit, coupling between the qubit states should be maximally inhibited, to allow for longer entanglement lifetime, and (ii) one has to take into account the fact that performing quantum operations on the qubits will result in faster decay of entanglement.

#### IV. NON-MARKOVIAN EFFECTS

In the Markovian approximation, the evolution of the system does not depend on its history. This approximation applies when the correlation times of the thermal bath are much smaller than any time scale associated with the system under consideration. However, there are cases where the bath reacts to the dynamics of the system over a certain memory time  $\tau_M$ , and the Markov approximation is thus not valid. In such instances, the evolution of the system would depend on its history [26]. Non-Markovian effects on the entanglement dynamics have been extensively studied in recent years [12–14].

In order to include non-Markovian effects, one has to include the history of the system. In the simplest approximation [26], this adds up to a form of the quantum master equation

$$\dot{\rho}(t) = -i[\mathcal{H}, \rho] + \int_0^t K(t-t') \mathcal{L}\rho(t') dt', \quad (11)$$

where the memory kernel  $K(t)$  defines the response of the bath to the history of the system. Note that not every form of the memory kernel is possible, as positivity of the density matrix may be lost [32].

One can now repeat the procedure described above, and rewrite Eq. (11) in a vector form, which yields the integro-differential vector equation

$$\dot{\tilde{\rho}}(t) = \int_0^t \hat{M}(t-t') \tilde{\rho}(t') dt'. \quad (12)$$

The time dependence of the  $\hat{M}$  matrix is such that for elements derived from the Lindblad superoperator one attaches the kernel  $K(t-t')$ , and for the elements derived from the Hamiltonian one attaches a  $\delta$  function,  $\delta(t-t')$ . One can now Laplace-transform Eq. (12) and obtain an algebraic vector equation  $s\tilde{\rho}(s) - \tilde{\rho}(0) = \hat{M}(s)\tilde{\rho}(s)$ , where  $s$  is the (complex) Laplace variable. Note that the Laplace transform  $\hat{M}(s)$  has a simple form, as the Hamiltonian elements are multiplied by unity (which is the transform for the  $\delta$  function) and the Lindbladian elements are multiplied by the Laplace transform of  $K(t)$ . The formal solution of the above equation is thus given by

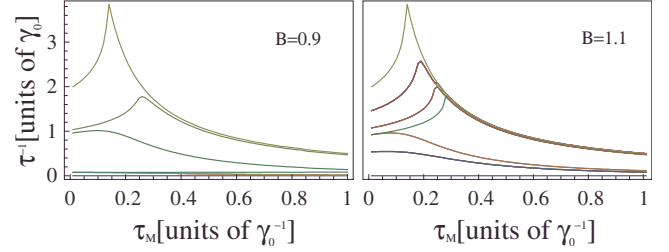


FIG. 5. (Color online) Relaxation rates (at  $T=0$ ,  $B/J=0.9, 0.1$ ) as a function of the memory time  $\tau_M$ . Note the resonant structure of the time scales, and the bunching of different time scales at large  $\tau$ .

$$\rho(t) = \int_{-i\infty}^{i\infty} e^{st} [s\mathbb{I} - \hat{M}(s)]^{-1} \rho(0) ds, \quad (13)$$

where  $\mathbb{I}$  is the unit matrix. From Eq. (13) it can be seen that if the secular equation  $\det[s\mathbb{I} - \hat{M}(s)]$  has solutions, the real part of these solutions defines a relaxation time scale (via a Cauchy-like integration over poles). One can thus obtain the relaxation times from a numerical solution of the secular equation, without the need to solve the full non-Markovian dynamics.

In the simplest approximation [32–35] the memory kernel is given by an exponential form,

$$K(t) = \tau_M^{-1} \exp(-|t|/\tau_M), \quad (14)$$

where  $\tau_M$  is the memory time. The Laplace transform is thus  $K(s) = 1/(\tau_M s + 1)$ , which yields a polynomial secular equation. Note that in the limit  $\tau_M \rightarrow 0$  the Markovian limit is exactly obtained. We have solved this equation numerically, and found that it always has  $N-1$  solutions with negative real part and a single solution with  $s=0$ , corresponding to the steady state. This means that, for an exponential memory kernel, one can identify different processes that have different relaxation rates. We note that, if one takes a more complicated memory kernel (say a power law), then the secular equation becomes transcendental, with no simple poles. In that kind of environment, one cannot simply attach different time scales to different processes, and the full dynamics of the system must be calculated [36].

The above scheme can now be applied to our two-qubit model. In Fig. 5 the inverse time scales are plotted as functions of the memory time  $\tau_M$  for  $B/J=0.9, 1.1$ . We find that the time scales have a resonant structure as a function of  $\tau_M$ , and in fact exhibit a cusp at the resonance. In addition, at relatively large  $\tau_M$ , all the different time scales merge into a single group, i.e., there is no longer a separation of the time scale for the relaxation processes of different coherent states.

The long-memory-time behavior may be understood by the fact that in this case the bath memory dominates the relaxation processes, giving rise to a single time scale. The longer the memory time, the more weight is given in the relaxation process to states far from the steady state, and hence the relaxation rate diminishes. However, the rise in relaxation rate at small memory times is a surprising effect, which comes about due to the complex nature of the interaction between the two-qubit system and the environment.

It will be interesting to examine the effect of non-Markovian dynamics on the entanglement (i.e., on the long-time behavior of the concurrence). For instance, the authors of Ref. [12] find that, in the absence of interaction between the qubits, the concurrence may be revived due to the non-Markovian bath. Unfortunately, the formalism presented in this section does not allow for an evaluation of the asymptotic behavior of the concurrence, and a full solution of the integro-differential master equation is needed. However, from Fig. 5 it is clear that all the relaxation rates, and specifically that associated with relaxation of  $\rho_{14}$ , decrease with increasing  $\tau_M$ . This means that the concurrence, which in our system is directly associated with  $\rho_{14}$ , should persist for longer times, i.e., non-Markovian effects contribute to the entanglement.

### V. SUMMARY

In this paper we have studied the relaxation dynamics of a model interacting two-qubit system in the presence of an Ohmic thermal environment. The qubits were modeled as spin- $\frac{1}{2}$  particles with spin-spin coupling and in the presence of a magnetic field. The environment was limited to induce only single spin-flip events. Within this model we analytically calculated the relaxation rates of different processes. Our main result is that disentanglement may be critically slowed down in the  $T \rightarrow 0$  limit by varying the magnetic field, and entanglement sudden death may be completely avoided (or at least exponentially suppressed for low temperatures). This was explicitly shown by calculating the concurrence dynamics of a highly entangled state for different magnetic fields. We have also shown that a transverse magnetic field may destroy this effect.

In addition, we have introduced a simple way to include non-Markovian effects in the calculation of the relaxation rates. We have shown that the different relaxation rates ex-

hibit an interesting nonmonotonic dependence on the environment memory time, with a cusp-like resonance. For long memory times, we have found that the different time scales merge into a single time scale.

In order to experimentally verify the results presented here, one needs an experiment with a well-controlled two-qubit system, where the energy levels can be controlled. A promising candidate is a system composed of two coupled two-quantum-dot qubits [37], where the qubit levels may be controlled by external gates, and a high level of control of the qubit states has already been demonstrated experimentally [38].

### ACKNOWLEDGMENTS

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### APPENDIX: THE $\hat{M}$ MATRIX

In this appendix we give the full form of the matrix  $\hat{M}$ . The starting point is the Lindblad equation, Eq. (2), into which the full form of the relaxation operators [i.e., Eqs. (5) and (6)] has been inserted. The resulting Lindblad operator is then cast into the linear form [27], resulting in the  $16 \times 16$   $\hat{M}$  matrix, which is given by the block-diagonal form

$$\hat{M} = \begin{pmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & M_3 & 0 \\ 0 & 0 & 0 & M_4 \end{pmatrix}, \quad (\text{A1})$$

where the zeros stand for  $4 \times 4$  zero matrices, and the  $M_i$  are  $4 \times 4$  matrices given by

$$M_1 = \begin{pmatrix} -\frac{2\gamma_0}{1+e^{(B+J)/T}} & \frac{e^{(B+J)/T}\gamma_0}{1+e^{(B+J)/T}} & \frac{e^{(B+J)/T}\gamma_0}{1+e^{(B+J)/T}} & 0 \\ \frac{\gamma_0}{1+e^{(B+J)/T}} & \left(\frac{1}{1+e^{(B+J)/T}} - 2 + \frac{1}{1+e^{(J-B)/T}}\right)\gamma_0 & 0 & \frac{e^{B/T}\gamma_0}{e^{B/T}+e^{J/T}} \\ \frac{\gamma_0}{1+e^{(B+J)/T}} & 0 & \left(\frac{1}{1+e^{(B+J)/T}} - 2 + \frac{1}{1+e^{(J-B)/T}}\right)\gamma_0 & \frac{e^{B/T}\gamma_0}{e^{B/T}+e^{J/T}} \\ 0 & \frac{e^{J/T}\gamma_0}{e^{B/T}+e^{J/T}} & \frac{e^{J/T}\gamma_0}{e^{B/T}+e^{J/T}} & -\frac{2e^{B/T}\gamma_0}{e^{B/T}+e^{J/T}} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \gamma_1 & -B-J & 0 & 0 \\ B+J & \gamma_1 & 0 & 0 \\ 0 & 0 & \gamma_1 & -B-J \\ 0 & 0 & B+J & \gamma_1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} \gamma_2 & -2B & 0 & 0 \\ 2B & \gamma_2 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 \\ 0 & 0 & 0 & \gamma_3 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} \gamma_4 & J-B & 0 & 0 \\ B-J & \gamma_4 & 0 & 0 \\ 0 & 0 & \gamma_4 & J-B \\ 0 & 0 & B-J & \gamma_4 \end{pmatrix}, \quad (A2)$$

with the parameters  $\gamma_i$  given by

$$\gamma_1 = \frac{\{\sinh(B/T) - 2[\cosh(B/T) + \cosh(J/T)]\}\gamma_0}{2[\cosh(B/T) + \cosh(J/T)]},$$

$$\gamma_2 = \left( -\frac{1}{1 + e^{(B+J)/T}} - \frac{1}{1 + e^{(J-B)/T}} \right) \gamma_0,$$

$$\gamma_3 = \left( \frac{1}{1 + e^{(B+J)/T}} - 1 - \frac{1}{1 + e^{(B-J)/T}} \right) \gamma_0,$$

$$\gamma_4 = -\frac{2[\cosh(B/T) + \cosh(J/T)] + \sinh(B/T)}{2[\cosh(B/T) + \cosh(J/T)]}. \quad (A3)$$

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